Asymptotic treatment of non-classically damped linear systems

W Fang, J-G Tseng, and JA Wickert

Department of Mechanical Engineering, Carnegie Mellon University, Pittsburgh PA 15213 USA

The presence of non-classical dissipation in a general discrete dynamic system is investigated through a perturbation method for the eigenvalues and vectors. Results accurate to second-order are obtained, with corrections to the base solution being expressed in terms of readily-calculated quadratic forms. Exact solutions, and the derived asymptotic ones, are compared with the predictions of the so-called method of approximate decoupling, in which certain non-classical dissipative terms are omitted from calculations in the eigenvalue problem. The perturbation method is discussed through its application in several examples, indicating circumstances in which a non-classically damped system can be well-approximated by an "equivalent" classically damped one. Somewhat surprisingly, the addition of non-classical damping does not necessarily increase the stability of all vibration modes, and the perturbation method is shown to be useful in identifying those critical modes.

1 Introduction

A linear dynamic system is said to have *classical normal modes* (CNM) if it possesses a complete set of real orthonormal eigenvectors. In general, an undamped system of the form (2) below will always possess CNM if certain well-established conditions are satisfied by the system matrices. However, when dissipative forces are present, CNM may or may not be present. Caughey and O'Kelly (1965) established the necessary and sufficient condition for existence of CNM in a damped linear system of the general form (1). Also worthwhile to note is the possibility that some modes will remain classical, and others will not, in the presence of non-classical dissipation (Felszeghy, 1989).

If Caughey and O'Kelly's criterion is not met, so that non-classical dissipation is present, treatment of free and forced response becomes substantially more complicated than in the case of classical damping. This is particularly so because the eigensolutions become complex, and orthogonality (or bi-orthogonality) relations among the eigenvectors are more difficult to establish (Foss, 1958; Vigneron, 1986).

Despite their accuracy, such exact methods have two primary disadvantages: they require significant numerical effort to determine the eigensolutions, and little physical insight is afforded by methods that are purely numerical. Specifically, exact methods based on such state space formulations as (6) below do not make evident the effects of non-classical damping on the eigenvalues and modes.

In treating non-classically damped linear systems, one approach is to ignore those damping terms that are non-classical, and retain the others. This approximation is termed the *method of approximate decoupling* (MAD), and error bounds for it have been discussed in (Shahruz and Ma, 1988; Shahruz, 1990; Felszeghy, 1993). More specifically, if a system is subject to non-classical damping, how do it's eigensolutions differ from those of a similar, analogous, system that is classically damped? To what extent can a non-classically damped system be approximated by a companion classically damped one? These questions are addressed in the present investigation.

2 Eigenvalue problem

Consider the free vibration of a discrete (or discretized) system described by the vector equation of motion

$$M\ddot{x} + C_1\dot{x} + Kx = 0 \tag{1}$$

where M, C_1, and K are real, symmetric, positive-definite, $N \times N$-dimensional matrices which operate on the generalized coordinates x. As a result of these conditions, it is always possible to simultaneously diagonalize M and K by a real similarity transformation. The required transformation is associated with the reduced (undamped) form

$$M\ddot{x} + Kx = 0 \ . \tag{2}$$

Let θ denote the normalized modal transformation that simultaneously diagonalizes M and K, and transform Eq. (1) into the normalized coordinates z. With the relations

$$x = \theta z, \quad \theta^T M\theta = I, \quad \theta^T C_1\theta = C_2, \quad \theta^T K\theta = \Lambda^k \ , \tag{3}$$

Eq. (1) becomes

$$I\ddot{z} + C_2\dot{z} + \Lambda^k z = 0 \tag{4}$$

where I is the identity matrix, and the superscript T denotes transposition. The derived damping matrix C_2 is also symmetric, and Λ^k becomes a diagonal matrix with real positive elements.

The second-order problem is recast into an equivalent state space format. With the definitions

$$A = \begin{bmatrix} I & 0 \\ 0 & -\Lambda^k \end{bmatrix}, \quad B_1 = \begin{bmatrix} C_2 & \Lambda^k \\ \Lambda^k & 0 \end{bmatrix}, \quad y = \begin{pmatrix} \dot{z} \\ z \end{pmatrix} \tag{5}$$

Eq. (4) becomes

$$A\dot{y} + B_1 y = 0 \ . \tag{6}$$

Synchronous solutions to Eq. (6) of the form $y = ue^{\lambda t}$ lead to the eigenvalue problem (EVP)

$$\lambda A u + B_1 u = 0 \ . \tag{7}$$

Although A and B_1 are each symmetric, positive definiteness in the EVP is not assured, nor is the statement that the matrices can be simultaneously diagonalized through a real coordinate transformation valid. Solutions of Eq. (7), however, remain useful to the degree that they characterize the system dynamics, but a disadvantage of this formulation is that the properties of the eigensolutions are not known in advance (Vigneron, 1986).

In the MAD, the modal damping matrix C_2 is decomposed into

$$C_2 = \Lambda^c + C_3 \ , \tag{8}$$

where Λ^c is diagonal, and C_3 is symmetric but has zeroes on its main diagonal. In the light of Eq. (3), Λ^c contains all of the classical, proportional damping terms, namely those on the diagonal of C_2; likewise, C_3 contains all of the off-diagonal dissipative terms. When the problem (1) is proportionally damped, then $C_3 = 0$, and so C_3 can be interpreted as dissipative coupling among the CNM. In the MAD, C_3 is approximated as being zero.

In the present study, interest is focused on how the behavior of (1) is altered when the dissipation is non-classical. To this end, consider B_1 as comprising small non-classical dissipation in the structure

$$B_1 = B + \varepsilon b \tag{9}$$

$$B = \begin{bmatrix} \Lambda^c & \Lambda^k \\ \Lambda^k & 0 \end{bmatrix} \quad b = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}$$

where ε is a scaling factor for the non-classical damping matrix $C_3 = \varepsilon C$, and the norm of C is $\mathcal{O}(1)$. With this nomenclature, the standard form of the EVP becomes

$$\lambda A u + (B + \varepsilon b)u = 0 \ . \tag{10}$$

3 Asymptotic forms

Consider solutions to the non-classically damped (10) in terms of corrections to those of the classically-damped problem as expressed in the expansions

$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} \cdots \tag{11}$$

$$\lambda = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} \cdots \ . \tag{12}$$

Substitution of Eqs. (11)-(12) into Eq. (10) and retention of terms of $\mathcal{O}(\varepsilon^2)$ and lower yield the perturbed eigenvalue problems:

Zero order

$$\lambda^{(0)} A u^{(0)} + B u^{(0)} = 0 \tag{13}$$

First order

$$\lambda^{(1)} A u^{(0)} + \lambda^{(0)} A u^{(1)} + B u^{(0)} + b u^{(0)} = 0 \tag{14}$$

Second order

$$\lambda^{(0)}Au^{(2)} + \lambda^{(1)}Au^{(1)} + \lambda^{(2)}Au^{(0)} + Bu^{(2)} + bu^{(1)} = 0 \ . \tag{15}$$

In what follows, it is assumed that the $\lambda_n^{(0)}$ of Eq. (13) are distinct, and with this requirement, the expansions (11)-(12) will provide non-secular solutions. If there are repeated eigenvalues, the solutions will split in the presence of non-classical damping, and the form of the expansions (11)-(12) must be altered.

Substitution and retention of terms of $\mathcal{O}(\varepsilon^2)$ and lower yield the perturbed orthogonality relations

$$u_m^{(0)T}Au_n^{(0)} = \delta_{nm} \tag{16}$$

$$u_m^{(0)T}Au_n^{(1)} + u_m^{(1)T}Au_n^{(0)} = 0 \tag{17}$$

$$2u_m^{(0)T}Au_n^{(2)} + u_m^{(1)T}Au_n^{(1)} = 0 \tag{18}$$

$$u_m^{(0)T}Bu_n^{(0)} = -\lambda_n^{(0)}\delta_{nm} \tag{19}$$

$$u_m^{(0)T}[Bu_n^{(1)} + bu_n^{(0)}] + u_m^{(1)T}Bu_n^{(0)} = -\lambda_n^{(1)}\delta_{nm} \tag{20}$$

$$u_m^{(0)T}[Bu_n^{(2)} + bu_n^{(1)}] + u_m^{(1)T}[Bu_n^{(1)} + bu_n^{(0)}] + u_m^{(2)T}Bu_n^{(0)} = -\lambda_n^{(2)}\delta_{nm} \ . \tag{21}$$

Equations (13)-(15) and (16)-(21) define the EVP and orthogonality conditions for the perturbed, non-classically damped system (10), for which the solution is asymptotic to the exact solution of the original problem (1).

4 First- and second-order corrections

For a classically damped system, the submatrices comprising A and B_1 are diagonal. Let these diagonal matrices Λ^c and Λ^k be given by $\Lambda^c = \mathrm{Diag}(2\zeta_n\omega_n)$ and $\Lambda^k = \mathrm{Diag}(\omega_n^2)$, where the ζ_n and ω_n^2 are real and non-negative. This zero-order solution satisfies (12) and the associated orthogonality relations (16) and (19). The $\lambda^{(0)}$ are given by

$$\lambda_n^{(0)} = -\zeta_n\omega_n + i\omega_n\sqrt{1-\zeta_n^2} \ , \quad 1 \le n \le N \tag{22}$$

$$\lambda_n^{(0)} = -\zeta_{n-N}\omega_{n-N} - i\omega_{n-N}\sqrt{1-\zeta_{n-N}^2} \ , \quad N+1 \le n \le 2N$$

where $i = \sqrt{-1}$. The normalized vectors become

$$u_1^{(0)T} = \{\lambda_1^{(0)}p_1, 0, \cdots, p_1, 0, \cdots\} \tag{23}$$

$$u_n^{(0)T} = \{0, \cdots, \lambda_n^{(0)}p_n, 0, \cdots, p_n, 0, \cdots\} \ , \quad 1 < n \le N$$

$$u_{N+1}^{(0)T} = \{\lambda_{N+1}^{(0)}p_{N+1}, 0, \cdots, p_{N+1}, 0, \cdots\}$$

$$u_1^{(0)T} = \{\lambda_1^{(0)}p_1, 0, \cdots, p_1, 0, \cdots\} \ , \quad N+1 < n \le 2N \ .$$

In Eq. (23), if $1 \le n \le N$, then $u_n^{(0)}$ has non-zero entries only in the n^{th} and $(n+N)^{th}$ positions. Likewise, if $N+1 \le n \le 2N$, then $u_n^{(0)}$ has non-zero entries in the $(n-N)^{th}$ and n^{th} positions. The normalization constants p_n will be complex, and given by

$$p_n^2 = \frac{1}{\lambda_n^{(0)2} - \omega_n^2} \ . \tag{24}$$

First-order. The eigenvector correction $u_n^{(1)}$ is expanded in terms of the zero-order eigenvectors $u_n^{(0)}$ as

$$u_n^{(1)} = \sum_{k=1}^{2N} \alpha_{nk}^{(1)} u_k^{(0)} \tag{25}$$

where the $\alpha_{nm}^{(1)}$ are coefficients to be determined. There are $2N$ linearly independent vectors $u_n^{(0)}$; therefore, they form a complete basis. Substitution of Eq. (25) into Eq. (13) and use of the orthogonality relations (16), (19), and either (17) or (20) yield the first-order corrections

$$\lambda_n^{(1)} = -u_n^{(0)T}bu_n^{(0)} \tag{26}$$

$$\alpha_{nn}^{(1)} = 0 \tag{27}$$

$$\alpha_{mn}^{(1)} = \frac{u_n^{(0)T}bu_m^{(0)}}{\lambda_n^{(0)} - \lambda_m^{(0)}} \ , \quad n \ne m \ .$$

Second-order. As above, the eigenvector correction is expanded in terms of the basis $u_n^{(0)}$

$$u_n^{(2)} = \sum_{k=1}^{2N} \alpha_{nk}^{(2)} u_k^{(0)} \tag{28}$$

Substitution into Eq. (14), use of orthogonality (16), (19), and either (18) or (21) yields the second-order corrections

$$\lambda_n^{(2)} = \sum_{k=1, k\ne n}^{2N} \frac{(u_k^{(0)T}bu_n^{(0)})^2}{\lambda_n^{(0)} - \lambda_k^{(0)}} \tag{29}$$

$$\alpha_{nn}^{(2)} = -\frac{1}{2}\sum_{k=1}^{2N}(\alpha_{nk}^{(1)})^2 \tag{30}$$

$$\alpha_{mn}^{(2)} = \frac{1}{\lambda_m^{(0)} - \lambda_n^{(0)}}(\lambda_m^{(1)}\alpha_{mn}^{(1)} + \sum_{k=1}^{2N}\alpha_{mk}^{(1)}u_n^{(0)T}bu_k^{(0)})$$

for $m \ne n$. Equations (22)-(30) in conjunction with the expansions (11) give the solution, accurate to second order, of the perturbed eigenvalue problem.

5 Discussion and application

The general method for solving a problem of the form (1) using the derived perturbation method is outlined as follows:

- Modal analysis on the reduced system (2) provides Λ^k and C_2,
- C_2 is decomposed into classical and non-classical terms in accordance with Eq. (8),
- Matrices A, B, and b as defined in Eqs. (5) and (9) are constructed,
- The zero-order problem is solved by using Eqs. (22)-(24), and
- The first and second-order corrections are found by using Eqs. (25)-(30).

The asymptotic corrections are expressed entirely in terms of the classically damped eigensolutions. In addition, implementation of the perturbation solution requires only the calculation of such quadratic forms as $u_n^{(0)T} b u_m^{(0)}$. From a computational standpoint, therefore, these calculations are relatively inexpensive when compared to direct solution of the original complex EVP. In iterative solution techniques, the perturbation solutions can serve as useful initial estimates.

By hypothesis, b contains only the non-classical damping terms and has zeroes along its diagonal; by considering Eq. (26), the first-order correction is always zero. In a general sense, the MAD represents the exact eigenvalues of a non-classically damped system to within $\mathcal{O}(\varepsilon^2)$[1]. In a similar manner, there are first-order corrections to the vectors, but the correction $\alpha_{nm}^{(0)}$ remains always zero. Thus, the MAD estimates the main diagonal of the (complex) modal matrix to within $\mathcal{O}(\varepsilon^2)$, with off-diagonal error of $\mathcal{O}(\varepsilon)$.

Without a rigorous treatment of error, some conclusions can be reached as to the asymptotic solution's radius of convergence. To the degree that $\lambda^{(1)}$ and $\lambda^{(2)}$ are intended to represent small corrections to the base solution $\lambda^{(0)}$, Eqs. (26)-(27) and (29)-(30) demonstrate that for a good approximation, the condition

$$\mathcal{O}(|\lambda_n^{(0)} - \lambda_m^{(0)}|) \geq \mathcal{O}(u_n^{(0)T} b u_m^{(0)}) \qquad (31)$$

must be satisfied. When this condition is not met, the "small" corrections become, in fact, large, and the assumptions underlying the perturbation series are violated. The following claim is therefore made: when $|\lambda_n^{(0)} - \lambda_m^{(0)}|$ is not small in comparison with $u_n^{(0)T} b u_m^{(0)}$, the perturbation solution is a good approximation of the exact solution, and visa versa. This claim is discussed further in the context of the examples below.

[1] An exception is the degenerate case $N = 1$ discussed in Example 1.

5.1 One degree of freedom

Consider the case of $N = 1$ for which

$$\ddot{z} + (2\zeta\omega + 2\zeta_1\omega\varepsilon)\dot{z} + \omega^2 z = 0 \qquad (32)$$

and the exact solution for the eigenvalues is

$$\lambda = -(\zeta + \varepsilon\zeta_1)\omega + i\omega\sqrt{1 - (\zeta + \varepsilon\zeta_1)^2}\ . \qquad (33)$$

In this case, the perturbed vectors have no meaning, but to second order, the eigenvalue becomes

$$\lambda = -(\zeta + \varepsilon\zeta_1)\omega \qquad (34)$$
$$+ i\omega\left(\sqrt{1-\zeta^2} - \varepsilon\frac{\zeta\zeta_1}{\sqrt{1-\zeta^2}} - \varepsilon^2\frac{\zeta_1^2}{2(1-\zeta^2)^{3/2}}\right),$$

which is identical to that obtained by a direct Taylor series expansion of Eq. (33) in powers of ε.

The exact and approximate roots are compared over a wide range of $\varepsilon\zeta_1$ in Fig. 1. The real components of the two solutions are identical. The imaginary components agree well, even for remarkably large values of the damping perturbation $\varepsilon\zeta_1$. For instance, when $\varepsilon\zeta_1 = 3\zeta$, there is only an 11% error in the calculation of $\Im(\lambda)$ and no error in $\Re(\lambda)$.

5.2 Two degrees of freedom

Consider the dynamic system described by

$$c\Lambda^c = \begin{bmatrix} 2\zeta_1\omega_1 & 0 \\ 0 & 2\zeta_2\omega_2 \end{bmatrix}, \Lambda^k = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}, \qquad (35)$$
$$C = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}.$$

The exact characteristic equation is

$$c\lambda^4 + \lambda^3(2\omega_1\zeta_1 + 2\omega_2\zeta_2) \qquad (36)$$
$$+\lambda^2(4\omega_1\omega_2\zeta_1\zeta_2 + \omega_1^2 + \omega_2^2 - \varepsilon^2 c^2)$$
$$+\lambda(2\omega_1^2\omega_2\zeta_2 + 2\omega_1\omega_2^2\zeta_1) + \omega_1^2\omega_2^2 = 0\ ,$$

with eigenvalues found by numerically calculating the attendant roots. As an interesting side issue, evidently, the sign of c does not affect the eigenvalues, as demonstrated by the presence of only c^2 in Eq. (36).

Calculation of the perturbed eigenvalues is tedious but straightforward using the theory developed above. The results are

$$\lambda_1^{(0)} = -\zeta_1\omega_1 + i\omega_1\sqrt{1-\zeta_1)^2} \qquad (37)$$

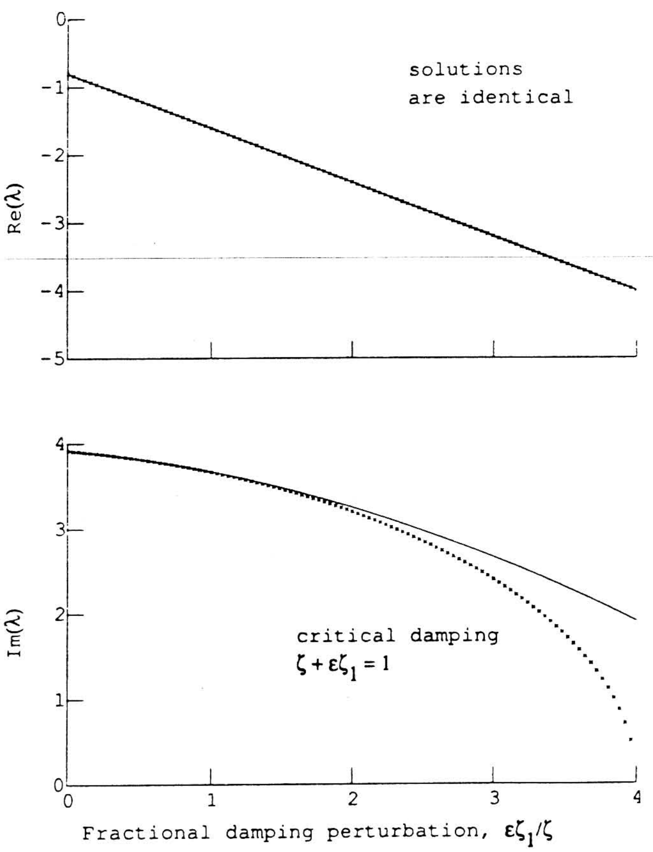

Figure 1: Comparison of the exact and perturbation solutions in Example 1. $\omega = 4$, $\zeta = 0.2$, and $\zeta_1 = 0.1$. Exact solution, × × ×; perturbation solution, — — —.

$$\lambda_2^{(0)} = -\zeta_2\omega_2 + i\omega_2\sqrt{1-\zeta_2)^2}\ , \qquad (38)$$

with $\lambda_1^{(1)} = \lambda_2^{(1)} = 0$. As for the second-order solution,

$$\lambda_1^{(2)} = 0 \qquad (39)$$

$$\Re(\lambda_1^{(2)}) = \frac{-1}{D}(c^2\omega_1\omega_2(\omega_1\zeta_2 - \omega_2\zeta_1))$$

$$\Im(\lambda_1^{(2)}) = \frac{-1}{D}\frac{c^2\omega_1(\omega_2(2\omega_1\zeta_1\zeta_2 - 2\omega_2\zeta_1^2 + \omega_2) - \omega_1^2)}{2\sqrt{1-\zeta_1^2}}$$

$$\lambda_2^{(2)} = 0 \qquad (40)$$

$$\Re(\lambda_2^{(2)}) = \frac{1}{D}(c^2\omega_1\omega_2(\omega_1\zeta_2 - \omega_2\zeta_1))$$

$$\Im(\lambda_2^{(2)}) = \frac{1}{D}\frac{c^2\omega_2(2\omega_1^2\zeta_2^2 - 2\omega_1\omega_2\zeta_1\zeta_2 + \omega_2^2 - \omega_1^2)}{2\sqrt{1-\zeta_2^2}}$$

where the denominator is given by

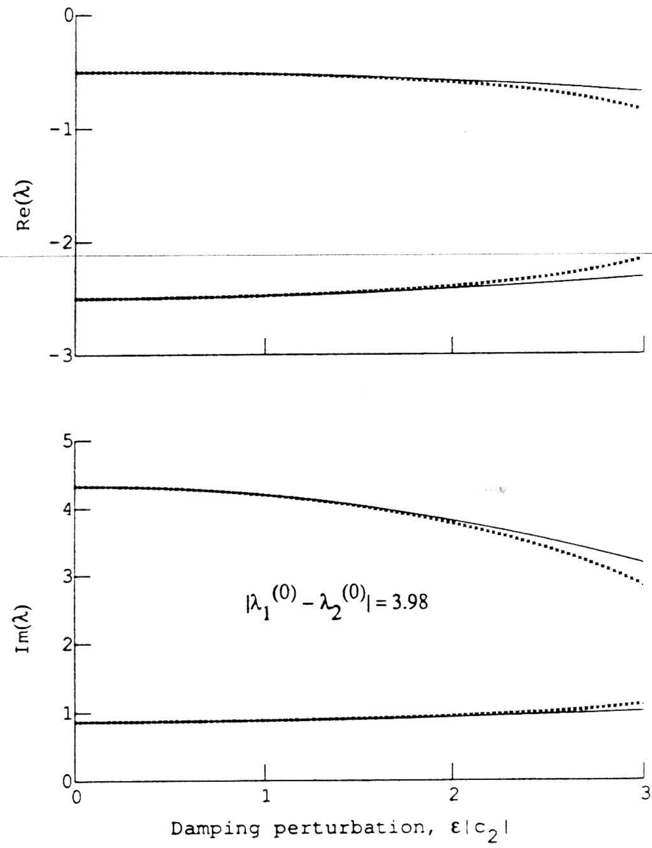

Figure 2: Comparison of the exact and perturbation solutions in Example 2. $\omega_1 = 1$, $\omega_2 = 5$, $\zeta_1 = 0.5$, $\zeta_2 = 0.5$, and $|c_2| = 0.5$. Exact solution, × × ×; perturbation solution, — — —.

$$\begin{aligned} D = {} & (2\omega_1\omega_2\sqrt{1-\zeta_1^2}\sqrt{1-\zeta_2^2} \\ & -2\omega_1\omega_2\zeta_1\zeta_2 + \omega_1^2 + \omega_2^2) \\ & \times(2\omega_1\omega_2\sqrt{1-\zeta_1^2}\sqrt{1-\zeta_2^2} \\ & +2\omega_1\omega_2\zeta_1\zeta_2 - \omega_1^2 - \omega_2^2)\ . \end{aligned} \qquad (41)$$

Figures 2 through 4 illustrate the comparison between the exact solution and the asymptotic one for several values of the system parameters ω_1, ω_2, ζ_1, and ${\zeta_2}^2$. Several points are noteworthy. As claimed above, the asymptotic solution performs best when $|\lambda_1^{(0)} - \lambda_2^{(0)}|$ is not small. When the difference is about 4, Fig. 2 demonstrates that the approximate solution is quite accurate even for values as large as $\varepsilon c = 3$. This corresponds to a non-classical damping ratio of 1.5, some 3

[2] The solution predicted by the MAD corresponds to the case $\varepsilon = 0$.

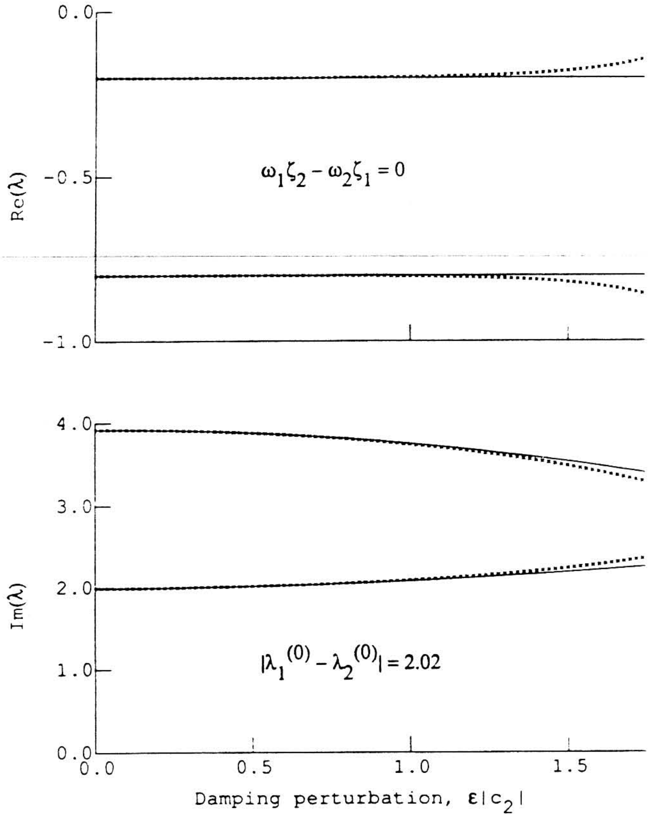

Figure 3: Comparison of the exact and perturbation solutions in Example 2. $\omega_1 = 2$, $\omega_2 = 4$, $\zeta_1 = 0.1$, $\zeta_2 = 0.2$, and $|c_2| = 0.5$. Exact solution, x x x; perturbation solution, — — —.

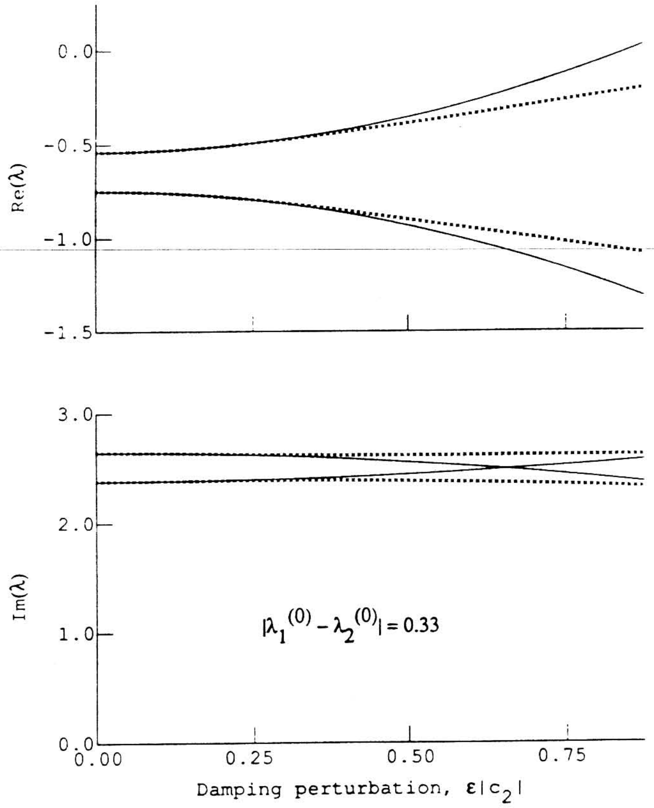

Figure 4: Comparison of the exact and perturbation solutions in Example 2. $\omega_1 = 2.5$, $\omega_2 = 2.7$, $\zeta_1 = 0.3$, $\zeta_2 = 0.2$, and $|c_2| = 0.5$. Exact solution, x x x; perturbation solution, — — —.

times the classical damping ratio. Thus, the perturbation method provides an accurate approximation even for a situation in which the non-classical effects dominate the classical ones. When the error parameter is small, as in Fig. 4, the range over which the perturbation solution is accurate becomes substantially more restricted.

As Fig. 3 and Eq. (39) show, when $\omega_1\zeta_2 - \omega_2\zeta_1 = 0$, the real components of the eigenvalues are remarkably insensitive to increases in the non-classical damping factor εc.

Somewhat surprisingly, the addition of non-classical damping does not increase the stability of both eigenvalues. In Figs. 2, 3, and 4, one $\Re(\lambda)$ locus becomes more negative with increasing εc, but its companion does not. Further, the perturbation solution predicts opposite concavity of the $\Re(\lambda)$ loci. This mode destabilization which results when the non-classical terms are large is a manifestation of the transition to negative definiteness of C_2 as ε is increased.

The asymptotic solution is also able to predict trends in the root loci, namely, their concavity. This attribute is absent in the MAD. From a design standpoint, it is useful to know the effect of non-classical damping on a system's stability. As demonstrated above, non-classical damping can either increase or decrease stability, with different behavior for different modes. Through perturbation, the critical modes which can be destabilized are identified. This capability is also absent from the MAD.

The exact and approximate eigenvectors are also calculated in this example. Table 1 demonstrates the comparison for one set of system parameters while εc is changed. There is a substantial phase shift between coordinates, even for small non-classical damping, and as expected, this shift vanishes when the system is classically damped. The perturbation solution appears to accurately capture the phase shift, but the MAD does miss it entirely. Additionally, in comparing the exact second mode vectors for the cases of $\varepsilon = 0$ and 3, the MAD predicts that the first coordinate has no displacement, but with $\varepsilon = 3$, it actually has displace-

εc_2	Exact		Perturbation	
	First mode	Second mode	First mode	Second mode
0.0	$\begin{Bmatrix} 1 \\ 0.0\,(0°) \end{Bmatrix}$	$\begin{Bmatrix} 0.0\,(0°) \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0.0\,(0°) \end{Bmatrix}$	$\begin{Bmatrix} 0.0\,(0°) \\ 1 \end{Bmatrix}$
0.5	$\begin{Bmatrix} 1 \\ 0.022\,(111°) \end{Bmatrix}$	$\begin{Bmatrix} 0.113\,(-111°) \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0.023\,(111°) \end{Bmatrix}$	$\begin{Bmatrix} 0.111\,(-111°) \\ 1 \end{Bmatrix}$
1.0	$\begin{Bmatrix} 1 \\ 0.046\,(112°) \end{Bmatrix}$	$\begin{Bmatrix} 0.231\,(-112°) \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0.045\,(112°) \end{Bmatrix}$	$\begin{Bmatrix} 0.218\,(-112°) \\ 1 \end{Bmatrix}$
1.5	$\begin{Bmatrix} 1 \\ 0.072\,(112°) \end{Bmatrix}$	$\begin{Bmatrix} 0.361\,(-112°) \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0.067\,(112°) \end{Bmatrix}$	$\begin{Bmatrix} 0.316\,(-111°) \\ 1 \end{Bmatrix}$
2.0	$\begin{Bmatrix} 1 \\ 0.103\,(113°) \end{Bmatrix}$	$\begin{Bmatrix} 0.514\,(-113°) \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0.088\,(113°) \end{Bmatrix}$	$\begin{Bmatrix} 0.402\,(-111°) \\ 1 \end{Bmatrix}$
2.5	$\begin{Bmatrix} 1 \\ 0.142\,(114°) \end{Bmatrix}$	$\begin{Bmatrix} 0.711\,(-114°) \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0.108\,(114°) \end{Bmatrix}$	$\begin{Bmatrix} 0.475\,(-111°) \\ 1 \end{Bmatrix}$
3.0	$\begin{Bmatrix} 1 \\ 0.204\,(117°) \end{Bmatrix}$	$\begin{Bmatrix} 1.02\,(-117°) \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0.128\,(115°) \end{Bmatrix}$	$\begin{Bmatrix} 0.533\,(-111°) \\ 1 \end{Bmatrix}$

Table 1: Comparison of the eigenvectors of the exact and perturbation solutions for Example 2. $\omega_1 = 1$, $\omega_2 = 5$, $\zeta_1 = 0.5$, $\zeta_2 = 0.5$, and $|c_2| = 0.5$. The case $\varepsilon = 0$ corresponds to the approximation of no off-diagonal damping terms.

ment greater than that of the second coordinate. In short, non-classical damping can significantly distort the mode shapes relative to the classically damped case, and the asymptotic solution appears to capture the character of these changes.

6 Summary

Non-classical damping in discrete (or discretized) models of dynamic systems is investigated through a derived, second-order, perturbation method for the eigenvalues and vectors. These asymptotic results are compared with the predictions of the commonly-used method of approximate decoupling. The MAD represents the eigenvalues of a non-classically damped system to within $\mathcal{O}(\varepsilon^2)$ in most cases, and estimates the main diagonal of the modal matrix to within $\mathcal{O}(\varepsilon^2)$, with off-diagonal error of $\mathcal{O}(\varepsilon)$. Through examples, other features of the MAD and the asymptotic solution are identified, including opposite concavity of the $\Re(\lambda)$ loci in ε.

Acknowledgment

This material is based upon work supported by the National Science Foundation and the author's group of industrial sponsors.

References

Caughey, T.K., and O'Kelly, M.E.J., 1965, "Classical Normal Modes in Damped Linear Dynamic Systems," *ASME Journal of Applied Mechanics*, Vol. 32, pp. 583-588.

Felszeghy, S.F., 1989, "The Development of Natural Modes of Free Vibration for Linear Discrete Systems From the Synchronous Motion Assumption," *ASME Journal of Vibration, Acoustics, Stress, and Reliability in Design*, Vol. 111, pp. 77-81.

Felszeghy, S.F., 1993, "On Uncoupling and Solving the Equations of Motion of Vibrating Linear Discrete Systems," *ASME Journal of Applied Mechanics*, Vol. 60, pp. 456-462.

Foss, K.A., 1958, "Coordinates Which Uncouple the Equations of Motion of Damped Linear Dynamic Systems," *ASME Journal of Applied Mechanics*, Vol. 25, pp. 361-364.

Shahruz, S.M., and Ma, F., 1988, "Approximate Decoupling of the Equations of Motion of Linear Underdamped Systems," *ASME Journal of Applied Mechanics*, Vol. 55, pp. 717-720.

Shahruz, S.M., 1990, "Approximate Decoupling of the Equations of Motion of Damped Linear Systems," *Journal of Sound and Vibration*, Vol. 136, pp. 51-64.

Vigneron, F.R., 1986, "A Natural Modes Model and Modal Identities for Damped Linear Structures," *ASME Journal of Applied Mechanics*, Vol. 53, pp. 33-38.